

ANTICOMMUTING INTEGRALS AND FERMIONIC FIELD THEORIES FOR TWO-DIMENSIONAL ISING MODELS †

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Abstract

We review the applications of the integral over anticommuting Grassmann variables (non-quantum fermionic fields) to the analytic solutions and the field-theoretical formulations for the 2D Ising models. The 2D Ising model partition function Q is presentable as the fermionic Gaussian integral. The use of the spin-polynomial interpretation of the 2D Ising problem is stressed, in particular. Starting with the spin-polynomial interpretation of the local Boltzmann weights, the Gaussian integral for Q appears in the universal form for a variety of lattices, including the standard rectangular, triangular, and hexagonal lattices, and with the minimal number of fermionic variables (two per site). The analytic solutions for the correspondent 2D Ising models then follow by passing to the momentum space on a lattice. The symmetries and the question on the location of critical point have an interesting interpretation within this spin-polynomial formulation of the problem. From the exact lattice theory we then pass to the continuum-limit field-theoretical interpretation of the 2D Ising models. The continuum theory captures all relevant features of the original models near T_c . The continuum limit corresponds to the low-momentum sector of the exact theory responsible for the critical-point singularities and the large-distance behaviour of correlations. The resulting field theory is the massive two-component Majorana theory, with mass vanishing at T_c . By doubling of fermions in the Majorana representation, we obtain as well the 2D Dirac field theory of charged fermions for 2D Ising models. The differences between particular 2D Ising lattices are merely adsorbed, in the field-theoretical formulation, in the definition of the effective mass.

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Abstract

We review some aspects of the anticommuting integrals as applied to the analytic solutions and the field-theoretical formulations for the 2D Ising models. We stress, in particular, the use of the spin-polynomial interpretation of the 2D Ising problem.

1. Introduction. A remarkable feature of the two-dimensional Ising model, first established within the transfer-matrix and combinatorial considerations, is that it can be reformulated as a free fermion field theory. It was recognized later on that the natural tool to analyze the 2D Ising models is the notion of the integral over Grassmann variables (nonquantum fermionic fields) introduced by Berezin [1-8]. A simple interpretation of the 2D Ising model based on the integration over anticommuting Grassmann variables and the mirror-ordered factorization principle for the density matrix was introduced in [7,8]. The partition function is expressible here as a fermionic Gaussian integral even for the most general inhomogeneous distribution of the coupling parameters [7]. This also gives a few line derivation of the Onsager result [7]. The Gaussian fermionic representations has been recently constructed as well for the inhomogeneous dimer problems [9]. The method [7,8] do not involve traditional transfer-matrix or combinatorial considerations. Schematically, we have:

$$Q = \operatorname{Sp}_{(\sigma)} Q(\sigma) \rightarrow \operatorname{Sp}_{(\sigma|a)} Q(\sigma|a) \rightarrow \operatorname{Sp}_{(a)} Q(a) = Q. \quad (1)$$

We start here with the original spin partition function Q , and then we introduce, in a special way, a set of new anticommuting Grassmann variables (a) , thus passing to a mixed $(\sigma|a)$ representation. Eliminating spin variables in this mixed representation, we obtain purely fermionic expression (fermionic Gaussian integral) for the same partition function Q . Grassmann variables are introduced, at the first stages, by factorization of the local weights. The factorization ideas resemble, in a sense, the idea of insertion of Dirac's unity, $\sum |a\rangle\langle a| = 1$, in transformations in quantum mechanics [10]. An important ingredient of the method is also the mirror-ordering procedure for the arising noncommuting Grassmann factors [7,8]. The original variant of the method implies two Grassmann variables *per bond* in the resulting fermionic integral, since we start here with the factorization of the bond Boltzmann weights [7]. An important modification was introduced in [8], where we start with the factorization of the *cell* weights presented by three-spin polynomials. This enables us to obtain the fermionic representation for Q with only two fermionic variables *per site* which provides essential simplifications in the analysis [8]. The symmetries and the criticality conditions have interesting spin-polynomial interpretation [8]. In this report we comment on the analytic results obtained so far within this

spin-polynomial fermionic approach and derive the effective field-theoretical (continuum-limit) formulations for the 2D Ising models valid near T_c . The resulting field theory is the massive free-fermion Majorana theory. The form of the action is universal, particular Ising lattices, like rectangular, triangular and hexagonal lattices, are specified merely by the definition of mass. By doubling of fermions in the Majorana representation, we obtain as well the 2D Dirac field interpretation for 2D Ising models near T_c .

2. Free-fermion representation and analytic results (lattice case).

The spin-polynomial interpretation of the 2D Ising models can be illustrated by an example of the triangular lattice [8]. Let us consider the triangular Ising lattice as a rectangular one with the additional diagonal interaction introduced in each lattice cell. Let mn be the lattice sites on the corresponding rectangular net, $m, n = 1, 2, 3, \dots, L$, where L is lattice length, $N = L^2$ is the number of lattice sites. At final stages we assume $L^2 \rightarrow \infty$. The hamiltonian of the triangular 2D Ising model is [8]:

$$-\beta H(\sigma) = \sum_{mn} [b_1 \sigma_{mn} \sigma_{m+1n} + b_2 \sigma_{m+1n} \sigma_{m+1n+1} + b_3 \sigma_{mn} \sigma_{m+1n+1}], \quad \beta = 1/kT, \quad (2)$$

where $\sigma_{mn} = \pm 1$ are the Ising spins, $b_\alpha = J_\alpha/kT$ are dimensionless coupling constants, J_α are the exchange energies, kT is temperature in energy units. In what follows we assume the purely ferromagnetic case, $b_\alpha > 0$. Respectively, the partition function and the free energy per site are given as follows:

$$Z = \sum_{(\sigma)} e^{-\beta H(\sigma)}, \quad -\beta f_Z = \lim_{N \rightarrow \infty} \frac{1}{N} \ln Z, \quad (3)$$

where the sum is taken over 2^N spin configurations provided by $\sigma_{mn} = \pm 1$ at each site. Noting the identity for the typical bond weight $e^{b\sigma\sigma'} = \cosh b + \sinh b \cdot \sigma\sigma'$, which readily follows from $(\sigma\sigma')^2 = +1$, we obtain:

$$Z = (2 \cosh b_1 \cosh b_2 \cosh b_3)^N Q, \quad N = L^2 \rightarrow \infty, \quad (4)$$

where Q is the reduced partition function [8]:

$$Q = \text{Sp}_{(\sigma)} \left\{ \prod_{mn} (1 + t_1 \sigma_{mn} \sigma_{m+1n}) (1 + t_2 \sigma_{m+1n} \sigma_{m+1n+1}) (1 + t_3 \sigma_{mn} \sigma_{m+1n+1}) \right\}, \quad (5)$$

where $t_\alpha = \tanh b_\alpha$, and $Sp_{(\sigma)} = 2^{-N} \sum_{(\sigma)}$ is the normalized spin averaging.

The spin-polynomial interpretation arises if we multiply the three bond weights forming a triangular cell in Q . Noting that $\sigma_1 \sigma_2 \cdot \sigma_2 \sigma_3 = \sigma_1 \sigma_3$, etc, we obtain [8]:

$$Q = \text{Sp}_{(\sigma)} \prod_{mn} (\alpha_0 + \alpha_1 \sigma_{mn} \sigma_{m+1n} + \alpha_2 \sigma_{m+1n} \sigma_{m+1n+1} + \alpha_3 \sigma_{mn} \sigma_{m+1n+1}), \quad (6)$$

where for the triangular lattice we have $\alpha_0 = 1 + t_1 t_2 t_3$, $\alpha_1 = t_1 + t_2 t_3$, $\alpha_2 = t_2 + t_1 t_3$, $\alpha_3 = t_3 + t_1 t_2$, the case of the rectangular lattice follows with $t_3 = 0$. As is discussed in [8] the hexagonal and some other lattices are covered by (6) as well, with the correspondent specification of the α parameters. In what follows we consider the three-spin polynomial partition function Q , assuming that $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are some numerical parameters.

The partition function (6) can be transformed into the fermionic Gaussian integral as follows [8]:

$$Q = \int \prod_{mn} dc_{mn}^* dc_{mn} \exp \left\{ \sum_{mn} [(\alpha_0 c_{mn} c_{mn}^* - \alpha_1 c_{mn} c_{m-1n}^* - \alpha_2 c_{mn} c_{mn-1}^* - \alpha_3 c_{mn} c_{m-1n-1}^*) - \alpha_1 c_{mn} c_{m-1n} - \alpha_2 c_{mn}^* c_{mn-1}^*] \right\}, \quad (7)$$

where c_{mn}, c_{mn}^* is a set of purely anticommuting Grassmann variables. The free-fermion representation for Q given in (7) is exact and completely equivalent to the original spin representation (6) up to the boundary effects negligible in the limit of infinite lattice $L^2 \rightarrow \infty$. [We do not comment here neither on the properties of the Grassmann variables, nor on the method of the derivation of this representation. The derivation is simple and takes one page of transformations, neglecting preliminary preparations, the details are to be seen in [8]. Also see the comments on the fermionic Gaussian integrals in [9].]

The explicit evaluation of the integral (7) can be performed by passing to the momentum space. For the squared partition function we find [8]:

$$Q^2 = \prod_{p=0}^{L-1} \prod_{q=0}^{L-1} [(\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2) - 2(\alpha_0 \alpha_1 - \alpha_2 \alpha_3) \cos \frac{2\pi p}{L} - 2(\alpha_0 \alpha_2 - \alpha_1 \alpha_3) \cos \frac{2\pi q}{L} - 2(\alpha_0 \alpha_3 - \alpha_1 \alpha_2) \cos \frac{2\pi(p+q)}{L}]. \quad (8)$$

Respectively, for the free energy we obtain:

$$\begin{aligned} -\beta f_Q &= \lim_{L \rightarrow \infty} \left(\frac{1}{L^2} \ln Q \right) = \\ &= \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \frac{dp}{2\pi} \frac{dq}{2\pi} \ln [(\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2) - 2(\alpha_0 \alpha_1 - \alpha_2 \alpha_3) \cos p - \\ &\quad - 2(\alpha_0 \alpha_2 - \alpha_1 \alpha_3) \cos q - 2(\alpha_0 \alpha_3 - \alpha_1 \alpha_2) \cos(p+q)]. \end{aligned} \quad (9)$$

It is easy to check that the free-energy form (9) yields the known exact expressions for the free energies of the rectangular, triangular, and hexagonal Ising lattices, with the correspondent specifications of the parameters α_j .

3. The symmetries and the critical point.

The symmetries provided by the above exact solution and the question on the location of the critical point have an interesting interpretation in the spin-polynomial language [8]. Introducing the notation $(\sigma_1, \sigma_2, \sigma_3)_{mn} \rightarrow (\sigma_{mn}, \sigma_{m+1n}, \sigma_{m+1n+1})$, the cell weight in the density matrix in (6) is given as the three-spin polynomial:

$$P_{123}(\sigma) = \alpha_0 + \alpha_1 \sigma_1 \sigma_2 + \alpha_2 \sigma_2 \sigma_3 + \alpha_3 \sigma_1 \sigma_3. \quad (10)$$

It is also useful to introduce the associated three-spin polynomial:

$$F_{123}(\sigma) = \alpha_0 - \alpha_1 \sigma_1 \sigma_2 - \alpha_2 \sigma_2 \sigma_3 - \alpha_3 \sigma_1 \sigma_3, \quad (11)$$

and we note the following interesting identity [8]:

$$\begin{aligned} (F_{123}(\sigma))^2 &= (\alpha_0 - \alpha_1 \sigma_1 \sigma_2 - \alpha_2 \sigma_2 \sigma_3 - \alpha_3 \sigma_1 \sigma_3)^2 = \\ &= (\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2) - 2(\alpha_0 \alpha_1 - \alpha_2 \alpha_3) \sigma_1 \sigma_2 - \\ &\quad - 2(\alpha_0 \alpha_2 - \alpha_1 \alpha_3) \sigma_2 \sigma_3 - 2(\alpha_0 \alpha_3 - \alpha_1 \alpha_2) \sigma_1 \sigma_3. \end{aligned} \quad (12)$$

It is seen that the combinations of the α -parameters occurring in $(F_{123})^2$ are exactly the same as in the momentum modes $Q^2(p|q)$ in (9):

$$Q^2(p|q) = (\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2) - 2(\alpha_0\alpha_1 - \alpha_2\alpha_3) \cos p - 2(\alpha_0\alpha_2 - \alpha_1\alpha_3) \cos q - 2(\alpha_0\alpha_3 - \alpha_1\alpha_2) \cos(p+q), \quad (13)$$

where $0 \leq p, q \leq 2\pi$, the limit $L^2 \rightarrow \infty$ is already assumed.

It appears that the following combinations of the α -parameters play important role in discussing the symmetries and the critical point [8]:

$$\begin{aligned} \alpha_0^* &= \frac{1}{2}(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3), & \bar{\alpha}_0 &= \frac{1}{2}(\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3), \\ \alpha_1^* &= \frac{1}{2}(\alpha_0 + \alpha_1 - \alpha_2 - \alpha_3), & \bar{\alpha}_1 &= \frac{1}{2}(\alpha_0 - \alpha_1 + \alpha_2 + \alpha_3), \\ \alpha_2^* &= \frac{1}{2}(\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3), & \bar{\alpha}_2 &= \frac{1}{2}(\alpha_0 + \alpha_1 - \alpha_2 + \alpha_3), \\ \alpha_3^* &= \frac{1}{2}(\alpha_0 - \alpha_1 - \alpha_2 + \alpha_3), & \bar{\alpha}_3 &= \frac{1}{2}(\alpha_0 + \alpha_1 + \alpha_2 - \alpha_3). \end{aligned} \quad (14)$$

The parameters α^* and $\bar{\alpha}$ are in fact the eigenvalues of the polynomials $\frac{1}{2}P_{123}$ and $\frac{1}{2}F_{123}$, respectively. By the "eigenvalues" we mean the four numbers which takes the polynomial as the spin variables run over their permissible values ± 1 . We note also the following identity:

$$\bar{\alpha}_0 \bar{\alpha}_1 \bar{\alpha}_2 \bar{\alpha}_3 = \alpha_0^* \alpha_1^* \alpha_2^* \alpha_3^* - \alpha_0 \alpha_1 \alpha_2 \alpha_3. \quad (15)$$

There are some evident symmetries in the solution. For instance, the free energy (9) is a symmetric function with respect to arbitrary permutations of $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ parameters. We can as well change the signs of any two of them, with $-\beta f_Q$ unchanged. There is also a less evident hidden symmetry in the solution, corresponding to the Kramers-Wannier duality in the case of the standard rectangular lattice. Namely, the partition function $Q\{\alpha\}$ is invariant under the transformation $\alpha_j \leftrightarrow \alpha_j^*$. This symmetry in fact holds already for the parameters of the separable fermionic modes (13), and can be proved making use of (12), see [8].

In order to establish the possible critical points, we have to look for zeroes of $Q^2(p|q)$ momentum modes. As it can be guessed already from the analogy between (12) and (13), there are four exceptional modes with $(p|q) = (0|0), (0|\pi), (\pi|0), (\pi|\pi)$. For these modes we have, respectively, $Q^2(p|q) = (2\bar{\alpha}_0)^2, (2\bar{\alpha}_1)^2, (2\bar{\alpha}_2)^2, (2\bar{\alpha}_3)^2$. We remember that the parameters α_j and hence $\bar{\alpha}_j$ are some functions of temperature. Thus, if at some temperature one of the above momentum modes vanishes, we fall at the point of phase transition. It can be shown that all other modes $Q^2(p|q)$ are definitely positive at all temperatures, there are no other critical points for physical values of the α -parameters. The possible criticality conditions can be combined into one equation:

$$\bar{\alpha}_0 \bar{\alpha}_1 \bar{\alpha}_2 \bar{\alpha}_3 = 0. \quad (16)$$

It can be shown also that if all bond interactions are ferromagnetic then $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3$ never become zero, and the critical point can only be associated with $\bar{\alpha}_0 = 0$, or $\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 = 0$. The criticality conditions with $\bar{\alpha}_k = 0, k = 1, 2, 3$, can only be realized when the antiferromagnetic interactions are involved in the hamiltonian.

If the critical point is associated with $\bar{\alpha}_j = 0$, then the singular part of the free energy (9) near T_c is given as follows [8]:

$$-\beta f_Q|_{\text{singular}} = \frac{(2\bar{\alpha}_j)^2}{16\pi\sqrt{(\alpha_0\alpha_1\alpha_2\alpha_3)_c}} \ln \frac{\text{const}}{(2\bar{\alpha}_j)^2}, \quad (17)$$

which implies that near T_c in order of magnitude $-\beta f_Q \sim \tau^2 \ln \tau^2$, where $\tau \sim |T - T_c|$. The specific heat have thus the log-type singularity, $C \sim |\ln \tau|$ as $T \rightarrow T_c$ (Onsager, 1944).

It is seen that the eigenvalues (14) play important role, but it is not so clear, in fact, how the role of the polynomial F_{123} can be understood in a less formal way, at the level of the original spin-variable formulation of the problem, prior to the analytic solution. Even more amusing grounds to be interesting in this game with the parameters α_j , $\bar{\alpha}_j$, α_j^* follow from the expression for the spontaneous magnetization below T_c . The 8-th power of the spontaneous magnetization $M = \langle \sigma_{mn} \rangle$ for model (6) is given by the following very simple expression [8]:

$$M^8 = (-1) \frac{\bar{\alpha}_0 \bar{\alpha}_1 \bar{\alpha}_2 \bar{\alpha}_3}{\alpha_0 \alpha_1 \alpha_2 \alpha_3} = 1 - \frac{\alpha_0^* \alpha_1^* \alpha_2^* \alpha_3^*}{\alpha_0 \alpha_1 \alpha_2 \alpha_3}. \quad (18)$$

This expression for M^8 holds true when the right hand side varies between 0 and 1, and $M^8 = 0$ otherwise. The well known expressions for the spontaneous magnetizations of the rectangular, triangular, and hexagonal lattices follow easily from (18) as particular cases. From (18) we find $M \sim \tau^{\frac{1}{8}}$ as $\tau \sim |T - T_c| \rightarrow 0$, with the universal value of the critical index $\beta = 1/8$ for the magnetization at the critical isobar. What are the hidden reasons for such simple expression for M^8 , is yet unknown.

4. Majorana field theory for the 2D Ising models.

The fermionic integral for Q is a suitable starting point to formulate the continuum field theory near T_c . We write once again the exact lattice action from (7) as follows:

$$\begin{aligned} S(c) = \sum_{mn} [& (\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3) c_{mn} c_{mn}^* + \alpha_1 c_{mn} (c_{mn}^* - c_{m-1n}^*) + \\ & + \alpha_2 c_{mn} (c_{mn}^* - c_{mn-1}^*) + \alpha_3 c_{mn} (c_{mn}^* - c_{m-1n-1}^*) + \\ & + \alpha_1 c_{mn} (c_{mn} - c_{m-1n}) + \alpha_2 c_{mn}^* (c_{mn}^* - c_{mn-1}^*)], \end{aligned} \quad (19)$$

with $c_{mn}^2 = c_{mn}^{*2} = 0$. Let us define lattice derivatives $\partial_m x_{mn} = x_{mn} - x_{m-1n}$, $\partial_n x_{mn} = x_{mn} - x_{mn-1}$, with $x_{mn} - x_{m-1n-1} = \partial_m x_{mn} + \partial_n x_{mn} - \partial_m \partial_n x_{mn}$. Introducing the new notation for the fermionic fields, c_{mn} , $c_{mn}^* \rightarrow \psi_{mn}$, $\bar{\psi}_{mn}$, from (19) we obtain:

$$\begin{aligned} S(\psi) = \sum_{mn} [& \underline{m} \psi_{mn} \bar{\psi}_{mn} + \lambda_1 \psi_{mn} \partial_m \bar{\psi}_{mn} + \lambda_2 \psi_{mn} \partial_n \bar{\psi}_{mn} - \\ & - \alpha_3 \bar{\psi}_{mn} \partial_m \partial_n \bar{\psi}_{mn} + \alpha_1 \psi_{mn} \partial_m \psi_{mn} + \alpha_2 \bar{\psi}_{mn} \partial_n \bar{\psi}_{mn}], \end{aligned} \quad (20)$$

$$\lambda_1 = \alpha_1 + \alpha_3, \quad \lambda_2 = \alpha_2 + \alpha_3, \quad \underline{m} = \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 = 2\bar{\alpha}_0.$$

It is easy to recognize in this still exact lattice action the typical field-theoretical like structure with mass term and kinetic part. Remember that we assume the ferromagnetic

case with the criticality condition $2\bar{\alpha}_0 = 0$. The parameter $\underline{m} = \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 = 2\bar{\alpha}_0$ play the role of mass in the field-theoretical interpretation, $\underline{m} \sim |T - T_c| \rightarrow 0$ as $T \rightarrow T_c$, the ordered phase corresponds to $\underline{m} < 0$.

Taking in (20) the formal limit to the continuum euclidean space, with $(mn) \rightarrow (x_1 | x_2)$, $\psi_{mn} \rightarrow \psi(x) = \psi\{x_1 | x_2\}$ and $\partial_m \rightarrow \partial_1 = \partial/\partial x_1$, $\partial_n \rightarrow \partial_2 = \partial/\partial x_2$, and neglecting the second-order $\partial_1 \partial_2$ kinetic term, we obtain the Majorana like fermionic action of the correspondent continuum theory:

$$S = \int d^2x [\underline{m} \psi(x) \bar{\psi}(x) + \lambda_1 \psi(x) \partial_1 \bar{\psi}(x) + \lambda_2 \psi(x) \partial_2 \bar{\psi}(x) + \alpha_1 \psi(x) \partial_1 \psi(x) + \alpha_2 \bar{\psi}(x) \partial_2 \bar{\psi}(x)], \quad (21)$$

where $\lambda_1, \lambda_2, \underline{m}$ are defined in (20). A nonstandard feature in this action is the presence of the "diagonal" kinetic terms like $\psi \partial_1 \bar{\psi}$, $\psi \partial_2 \bar{\psi}$. This terms can be eliminated by a suitable linear transformation of the fields $\psi, \bar{\psi}$. Omitting the details, after a suitable transformation of the fields and the d^2x space in (21), we come to the euclidean 2D Majorana action in canonical form:

$$S = \int d^2x [i \bar{m} \psi_1(x) \psi_2(x) + \psi_1(x) \partial_0 \psi_1(x) + \psi_2(x) \bar{\partial}_0 \psi_2(x)], \quad (22)$$

$$\partial_0 = \frac{1}{2}(\partial_1 - i\partial_2), \quad \bar{\partial}_0 = \frac{1}{2}(\partial_1 + i\partial_2), \quad \text{2D Majorana,}$$

with the rescaled mass:

$$\bar{m} = \frac{\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3}{(2\sqrt{(\alpha_0 \alpha_1 \alpha_2 \alpha_3)_c})^{1/2}}. \quad (23)$$

The 2D Ising model is presented in (22) as a field theory of free massive two-component Majorana fermions over the euclidean d^2x space. The Majorana fields ψ_1, ψ_2 in this representation are the linearly transformed fields $\psi, \bar{\psi}$ from (21). The axis of the d^2x space are also rescaled and rotated as we pass from (21) to (22). Respectively, the mass is rescaled according (23), here $()_c$ means the criticality condition $(\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3)_c = 0$. The Majorana field theory for the 2D Ising model, for the simplest case of the isotropic rectangular lattice, was constructed, by another method, in refs. [3,4]. The action (22) is a generalization of eq. (129) in [4], also see eqs. (2.11) in the first reference of [3]. The generalized action (22) captures the relevant features of the exact lattice 2DIM theories for the lattices covered by (6), see also (7), in the low-momentum sector, which is responsible for the large-distance behavior of the correlation functions and the thermodynamic singularities near T_c . In particular, the result (17) can still be recovered from (22).

5. The Dirac field interpretation of the 2D Ising models.

We can pass as well to the Dirac field theory of charged fermions by doubling the number of fermions in the Majorana representation (22). On formal level this corresponds to passing from Pfaffian like Gaussian integral to the determinantal Gaussian integral according the identity $|\text{Pfaff } A|^2 = \det A$, see [9]. To this end, we take two identical copies S' and S'' of (22) and consider the combined action $S_{\text{dirac}} = S' + S''$. In this action we introduce the new fermionic fields $\psi_1, \psi_2, \psi_1^*, \psi_2^*$ by substitution:

$$\psi_1' = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2^*), \quad i\psi_1'' = \frac{1}{\sqrt{2}}(\psi_1 - \psi_2^*), \quad \psi_2' = \frac{1}{\sqrt{2}}(\psi_2 + \psi_1^*), \quad i\psi_2'' = \frac{1}{\sqrt{2}}(\psi_2 - \psi_1^*). \quad (24)$$

In terms of new fields the action $S = S' + S''$ becomes:

$$S_{\text{dirac}} = \int d^2x \left[i\bar{m} (\psi_1(x) \psi_1^*(x) - \psi_2(x) \psi_2^*(x)) + \right. \\ \left. + \psi_1(x) (\partial_1 - i\partial_2) \psi_2^*(x) + \psi_2(x) (\partial_1 + i\partial_2) \psi_1^*(x) \right], \quad (25)$$

where the rescaled mass \bar{m} is given in (23). In matrix notation we find:

$$S = \int d^2x \left\{ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \left[i\bar{m} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \partial_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \partial_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] \begin{pmatrix} \psi_1^* \\ \psi_2^* \end{pmatrix} \right\}. \quad (26)$$

Introducing the 2D euclidean γ -matrices $\gamma_1 = \sigma_1$, $\gamma_2 = \sigma_2$ and $\gamma_5 = \gamma_1\gamma_2 = i\sigma_3$, where $\sigma_1, \sigma_2, \sigma_3$ are standard Pauli matrices, we obtain the same action in the γ -matrix interpretation:

$$S = \int d^2x \left[\Psi(x) (\bar{m}\gamma_5 + \gamma_1\partial_1 + \gamma_2\partial_2) \Psi^\dagger(x) \right] \quad (\text{euclidean 2D Dirac}), \quad (27)$$

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}, \quad \gamma_1^2 = \gamma_2^2 = 1, \quad \gamma_5 = \gamma_1\gamma_2, \quad \gamma_5^2 = -1,$$

where $\Psi = (\psi_1 | \psi_2)$, $\Psi^\dagger = (\psi_1^* | \psi_2^*)$ are the two-component charged spinors. We find here the euclidean action of the massive free 2D Dirac theory. The unconventional γ_5 matrix in the mass term can be eliminated, if one wants, by a redefinition of the fields, changing the sign of a one of the four spinor components.

We can pass as well to the 2D Minkovsky space. The 2D Minkovsky space is the complex plane. Hence we put $\{x_1 | x_2\} \rightarrow \{x | it\}$, $\Psi(x_1 | x_2) \rightarrow \Psi(x | it)$, $\partial_1 \rightarrow \partial_1 = \partial/\partial x$, $\partial_2 \rightarrow -i\partial_0 = -i\partial/\partial t$. From (25) or (26) we obtain:

$$S_{\text{dirac}} = i \int d^2x \Psi(x) (\sigma_3) [\bar{m} + \partial_1(\sigma_2) + \partial_0(i\sigma_1)] \Psi^\dagger(x). \quad (28)$$

Introducing the Minkovsky-space γ -matrices $\gamma^0 = \sigma_1$, $\gamma^1 = -i\sigma_2$, $\gamma^5 = \gamma^0\gamma^1 = \sigma_3$, and passing to new spinors $\bar{\Psi} = \Psi\gamma^5$, $\bar{\Psi}^\dagger = \Psi^\dagger$, we find:

$$S = i \int d^2x \bar{\Psi}(x) [\bar{m} + i\hat{\partial}] \bar{\Psi}^\dagger(x) \quad (\text{minkovsky 2D Dirac}), \quad (29)$$

$$\hat{\partial} = \gamma^\mu\partial_\mu = \gamma^0\partial_0 + \gamma^1\partial_1, \quad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \quad g^{\mu\nu} = \text{diag}(+|-).$$

We have here the 2D Dirac action in the Minkovsky space interpretation. This is to be compared with the action given in eq. (3.17) in [6] for a particular case of the rectangular lattice.

In conclusion, we have discussed the fermionic structure and related aspects of the 2D Ising models described by the three-spin polynomial partition function (6) and the equivalent Gaussian fermionic integral (7). The symmetries and the criticality conditions are most naturally expressed in terms of the parameters α , $\bar{\alpha}$, α^* arising in this spin-polynomial interpretation. Also, the spontaneous magnetization is expressible very simply in terms of these parameters, see (18), which property is not yet well understood. We then derived, directly from the exact lattice action, the continuum-limit Majorana and Dirac field theories for the models under discussion, including the standard rectangular,

triangular, and hexagonal 2DIM lattices. We find that the lattice parameters are merely absorbed, in the continuum limit, in the definition of mass. This confirms the universality ideas. Finally, we argue that even the standard rectangular lattice can be most easily analyzed, both in the exact lattice theory and in the field-theoretical approximation, starting with the spin-polynomial interpretation of the problem.

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